Applications of the Łojasiewicz–Simon Inequality

Based on lectures by Jiewon Park at the 2026 AMSI Winter School

These lectures study the Łojasiewicz–Simon inequality as a tool for proving convergence and uniqueness in geometric evolution problems. The basic difficulty is that monotonicity of an energy usually gives only square-integrability of the velocity and subsequential convergence to critical points; it does not by itself guarantee convergence to a unique limit. A Łojasiewicz inequality relates the energy deficit to the size of its gradient and thereby upgrades this weak information to a finite-length estimate.

We first examine the finite-dimensional theory and its extension to analytic functionals with Fredholm Hessian. We then apply the same underlying principle to harmonic map heat flow, mean curvature flow, and the asymptotic geometry of Ricci-flat manifolds. In each setting, compactness produces a possible limiting object, while a suitable Łojasiewicz-type inequality prevents the solution, rescaled flow, or geometry from drifting among different limits.

To create the final version of these notes, I (Steven Buchanan, human) took notes during the winter school lectures, and then fed them through ChatGPT for refinement.

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1.1 Finite-dimensional gradient flows

Let (M, g) be a Riemannian manifold and let $f : M \to \mathbb{R}$ be a smooth function that we wish to minimize.

Definition 1.1. The **gradient** of f is the vector field ∇f characterized by

$$Df_x(v) = g_x(\nabla f(x), v)$$

for every $x \in M$ and $v \in T_xM$.

The **negative gradient flow** of f is the equation

$$\frac{dx}{dt} = -\nabla f(x(t)).$$

Along a solution,

$$\frac{d}{dt} f(x(t)) = Df_{x(t)}(x'(t)) = -\left|\nabla f(x(t))\right|^2 = -\left|x'(t)\right|^2. \tag{1}$$

Thus the energy $f(x(t))$ is monotone nonincreasing. It either tends to $-\infty$ or converges to a finite limit

$$f_\infty \coloneqq \lim_{t\to\infty} f(x(t)).$$

If $f_\infty > -\infty$, then for each $T > 0$,

$$f(x(0)) - f(x(T)) = \int_0^T \left|x'(t)\right|^2 \, dt.$$

Letting $T \to \infty$ gives the energy identity

$$f(x(0)) - f_\infty = \int_0^\infty \left|x'(t)\right|^2 \, dt. \tag{2}$$

In particular, $x' \in L^2([0, \infty))$.

Proposition 1.2. Suppose that $f_\infty > -\infty$ and that the trajectory $\{x(t) : t \geq 0\}$ has compact closure in M. Then every accumulation point of the trajectory is a critical point of f.

Proof. Set
$$h(t) = |x'(t)|^2 = |\nabla f(x(t))|^2 .$$
The energy identity implies that $h \in L^1([0,\infty))$.

Along the flow,
$$h'(t) = -2 \operatorname{Hess} f_{x(t)}\big(\nabla f(x(t)), \nabla f(x(t))\big).$$
Because the trajectory has compact closure, both $\operatorname{Hess} f$ and ∇f are bounded along it. Hence h' is bounded, so h is uniformly continuous.

A nonnegative, uniformly continuous, integrable function on $[0,\infty)$ tends to zero. This standard result is sometimes called Barbalat's lemma. Therefore
$$\lim_{t\to\infty} |\nabla f(x(t))| = 0.$$

If
$$x_\infty = \lim_{j\to\infty} x(t_j)$$
for some sequence $t_j \to \infty$, then continuity of ∇f gives
$$\nabla f(x_\infty) = 0.$$

□

The L^2-bound on x' does not by itself imply that the trajectory converges. A stronger and very useful estimate is
$$\int_0^\infty |x'(t)|\, dt < \infty. \tag{3}$$
This says that the gradient-flow trajectory has finite length.

Indeed, for $t > s$,
$$d(x(t), x(s)) \leq \int_s^t |x'(r)|\, dr.$$
Thus a finite-length trajectory is Cauchy as $t \to \infty$. If the trajectory has compact closure, or if the ambient manifold is complete, it converges to a unique limit. By the preceding proposition, the limit is a critical point of f.

A central question is therefore: when can we prove (3)?

Example 1.3. Let
$$f(x) = \frac{1}{2}x^2.$$
Then
$$x'(t) = -x(t), \qquad x(t) = e^{-t}x(0),$$
and hence
$$f(x(t)) = \frac{1}{2}e^{-2t}x(0)^2.$$
In particular, the trajectory has finite length.

Example 1.4. For $m \in \mathbb{Z}_{\geq 2}$, let

$$f(x) = \frac{1}{2m} x^{2m}.$$

Then

$$x'(t) = -x(t)^{2m-1}.$$

If $x(0) \neq 0$, the solution is

$$x(t) = \frac{x(0)}{(1 + (2m-2)x(0)^{2m-2} t)^{1/(2m-2)}}.$$

If $x(0) = 0$, the solution is identically zero. In either case, the trajectory converges to zero and has finite length. The convergence is algebraic rather than exponential when $m \geq 2$.

Example 1.5. Suppose $\alpha \geq 0$ and

$$|x'(t)| = (1+t)^{-\alpha}.$$

Then

$$x' \in L^1([0,\infty)) \quad \Longleftrightarrow \quad \alpha > 1,$$

whereas

$$x' \in L^2([0,\infty)) \quad \Longleftrightarrow \quad \alpha > \frac{1}{2}.$$

Thus an L^2-bound on the speed need not imply finite length.

Proposition 1.6 (Finite-length consequence of a Łojasiewicz inequality)**.** Suppose

$$f_\infty = \lim_{t \to \infty} f(x(t)) > -\infty.$$

Assume that, for all sufficiently large t,

$$|f(x(t)) - f_\infty|^{1-\theta} \leq C \, |\nabla f(x(t))| , \qquad 0 < \theta \leq \frac{1}{2}. \tag{4}$$

Then

$$\int_0^\infty |x'(t)| \, dt < \infty.$$

Proof. Define the energy deficit

$$e(t) = f(x(t)) - f_\infty.$$

Because $f(x(t))$ decreases to f_∞, we have $e(t) \geq 0$, and

$$e'(t) = -\left|\nabla f(x(t))\right|^2 = -\left|x'(t)\right|^2.$$

Whenever $e(t) > 0$, the assumed inequality gives

$$\begin{aligned} -\frac{d}{dt} e(t)^\theta &= -\theta e(t)^{\theta-1} e'(t) \\ &= \theta e(t)^{\theta-1} \left|x'(t)\right|^2 \\ &\geq \frac{\theta}{C} e(t)^{\theta-1} e(t)^{1-\theta} \left|x'(t)\right| \\ &= \frac{\theta}{C} \left|x'(t)\right|. \end{aligned}$$

Integrating from a sufficiently large time T_0 to T gives

$$\int_{T_0}^{T} |x'(t)| \, dt \leq \frac{C}{\theta} \left(e(T_0)^\theta - e(T)^\theta \right) \leq \frac{C}{\theta} e(T_0)^\theta.$$

Letting $T \to \infty$ proves that the tail of the trajectory has finite length. The portion on $[0, T_0]$ has finite length by smoothness. □

Remark 1.7. The argument only requires the inequality along the trajectory for all sufficiently large times. In applications, one often proves a local Łojasiewicz inequality in a neighborhood of an accumulation point and then uses a separate trapping argument to show that the trajectory eventually remains in that neighborhood.

Remark 1.8. For a general smooth function, an inequality such as (4) need not hold. The classical finite-dimensional Łojasiewicz gradient inequality applies, in particular, to real-analytic functions near a critical point. Simon's theory extends this mechanism to suitable analytic functionals on infinite-dimensional spaces.

1.2 Infinite-dimensional gradient flows

Let X be a manifold of geometric objects, such as

- a space of functions $u : (M, g) \to \mathbb{R}$,
- a space of maps $u : M \to N$,
- a space of immersions $F : M \hookrightarrow \mathbb{R}^N$,
- a space of Riemannian metrics on M.

Let

$$\mathcal{E} : X \to \mathbb{R}$$

be an energy functional that we wish to minimize.

Suppose that X is equipped with a Riemannian metric. At $u \in X$, this gives an inner product G_u on the tangent space $T_u X$.

Definition 1.9 (Gradient and gradient flow)**.** The gradient $\nabla \mathcal{E}(u) \in T_u X$ is characterized by

$$D\mathcal{E}_u(V) = G_u(\nabla \mathcal{E}(u), V)$$

for every $V \in T_u X$.

Thus the gradient depends on the chosen metric G. The negative gradient flow is

$$\partial_t u = -\nabla \mathcal{E}(u).$$

Formally, along a sufficiently regular solution,

$$\frac{d}{dt} \mathcal{E}(u(t)) = - \left|\left| \nabla \mathcal{E}(u(t)) \right|\right|_G^2 = - \left|\left| \partial_t u(t) \right|\right|_G^2 . \tag{5}$$

We first examine several examples before discussing how an infinite-dimensional Łojasiewicz–Simon inequality can be proved.

1.3 Basic examples

Example 1.10 (The heat equation)**.** Let (M,g) be a closed Riemannian manifold and let $u \in C^\infty(M)$. Define

$$\mathcal{E}(u) = \frac{1}{2}\int_M |\nabla u|^2 \, d\mu_g,$$

and equip the space of functions with the L^2 metric

$$G_u(v,w) = \int_M vw \, d\mu_g.$$

This metric is independent of u.

For $v \in C^\infty(M)$, consider the variation

$$u_s = u + sv.$$

Then

$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=0} \mathcal{E}(u_s) &= \left.\frac{d}{ds}\right|_{s=0} \frac{1}{2}\int_M |\nabla(u+sv)|^2 \, d\mu_g \\ &= \int_M \langle \nabla u, \nabla v\rangle \, d\mu_g \\ &= \int_M (-\Delta u) v \, d\mu_g. \end{aligned}$$

Here we use the convention that $\Delta = \operatorname{div} \nabla$ is nonpositive on a closed manifold.

On the other hand,

$$\left.\frac{d}{ds}\right|_{s=0} \mathcal{E}(u_s) = G_u(\nabla\mathcal{E}(u), v) = \int_M \nabla\mathcal{E}(u)\, v \, d\mu_g.$$

Therefore

$$\nabla\mathcal{E}(u) = -\Delta u.$$

The negative gradient flow is consequently

$$\partial_t u = \Delta u.$$

Along the heat flow,

$$\frac{d}{dt}\mathcal{E}(u(t)) = -\int_M (\Delta u)^2 \, d\mu_g = -\int_M (\partial_t u)^2 \, d\mu_g.$$

The critical points are the functions satisfying

$$\Delta u = 0.$$

Since M is closed, these are precisely the functions that are constant on each connected component of M. Assume for simplicity that M is connected.

The critical points are therefore not isolated. Instead, they form the one-dimensional space of constant functions:

$$C^\infty(M) = \mathbb{R} \oplus \left\{ u \in C^\infty(M) : \int_M u \, d\mu_g = 0 \right\}.$$

The heat equation preserves the average

$$\bar{u} = \frac{1}{\mathrm{Vol}(M)} \int_M u(t)\, d\mu_g = \frac{1}{\mathrm{Vol}(M)} \int_M u(0)\, d\mu_g.$$

Set

$$w(t) = u(t) - \bar{u}.$$

Then w satisfies the heat equation and has zero average. The Poincaré inequality states that

$$\int_M |\nabla w|^2\, d\mu_g \geq \lambda_1 \int_M w^2\, d\mu_g,$$

where $\lambda_1 > 0$ is the first positive eigenvalue of $-\Delta$.

Therefore

$$\begin{aligned} \frac{d}{dt} ||w||_{L^2}^2 &= 2 \int_M w\, \partial_t w\, d\mu_g \\ &= 2 \int_M w\, \Delta w\, d\mu_g \\ &= -2 \int_M |\nabla w|^2\, d\mu_g \\ &\leq -2\lambda_1 ||w||_{L^2}^2 . \end{aligned}$$

Gronwall's inequality gives

$$||u(t) - \bar{u}||_{L^2} = ||w(t)||_{L^2} \leq e^{-\lambda_1 t} ||u(0) - \bar{u}||_{L^2} . \tag{6}$$

Thus the heat flow converges exponentially in L^2 to the average of the initial function.

Example 1.11 (Harmonic map heat flow)**.** Let

$$u : (M, g) \to (N, h)$$

be a smooth map between closed Riemannian manifolds. Define the Dirichlet energy by

$$\mathcal{E}(u) = \frac{1}{2} \int_M |du|^2\, d\mu_g = \int_M e(u)\, d\mu_g,$$

where

$$e(u) = \frac{1}{2} |du|^2$$

is the energy density.

Let

$$X = C^\infty(M, N).$$

There is a canonical identification

$$T_u X = \Gamma(u^* TN).$$

Indeed, if u_s is a curve of maps with $u_0 = u$, then

$$V(x) = \left. \frac{d}{ds} \right|_{s=0} u_s(x) \in T_{u(x)} N,$$

so V is a section of u^*TN.

Define the L^2 metric

$$G_u(V,W) = \int_M h(V,W)\,d\mu_g.$$

Consider a variation u_s and let

$$V = \left.\frac{d}{ds}\right|_{s=0} u_s.$$

If $\{e_i\}$ is a local orthonormal frame on M, then, using summation over i,

$$\begin{aligned}\left.\frac{d}{ds}\right|_{s=0} \mathcal{E}(u_s) &= \int_M \langle \nabla_s du_s(e_i), du_s(e_i)\rangle_h \; d\mu_g \\ &= \int_M \langle \nabla_i^u V, du(e_i)\rangle_h \; d\mu_g \\ &= -\int_M \langle \operatorname{tr}_g \nabla du, V\rangle_h \; d\mu_g.\end{aligned}$$

Definition 1.12. The **tension field** of u is

$$\tau(u) = \operatorname{tr}_g \nabla du = (\nabla du)(e_i, e_i).$$

A map is **harmonic** if

$$\tau(u) = 0.$$

The first-variation formula shows that

$$\nabla \mathcal{E}(u) = -\tau(u).$$

Therefore the negative gradient flow is

$$\partial_t u = \tau(u),$$

which is the **harmonic map heat flow**.

If $N \subset \mathbb{R}^L$ is isometrically embedded, then

$$\tau(u) = P_{T_u N} \Delta_g u,$$

where the Laplacian is applied componentwise and $P_{T_u N}$ denotes orthogonal projection onto $T_u N$. Equivalently, the normal component of $\Delta_g u$ can be expressed using the second fundamental form of the embedding. The signs in this formula depend on the convention for the second fundamental form.

For

$$N = S^{L-1} \subset \mathbb{R}^L,$$

one obtains the convention-independent formula

$$\tau(u) = \Delta_g u + |\nabla u|^2 \, u.$$

Along harmonic map heat flow,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(u(t)) &= -\int_M |\tau(u)|^2 \; d\mu_g \\ &= -\int_M |\partial_t u|^2 \; d\mu_g.\end{aligned}$$

Since $\mathcal{E} \geq 0$ and is nonincreasing, it has a finite limit $\mathcal{E}_\infty$. Consequently,

$$\mathcal{E}(u(0)) - \mathcal{E}_\infty = \int_0^\infty \int_M |\partial_t u|^2 \, d\mu_g \, dt.$$

Thus the speed is square-integrable in time. At this stage, however, we do not yet know whether

$$\int_0^\infty ||\partial_t u||_{L^2(M)} \, dt < \infty.$$

Remark 1.13. If ϕ is an isometry of (N, h) and u_* is harmonic, then $\phi \circ u_*$ is also harmonic. Thus target isometries can produce non-isolated families of critical points.

For the scalar heat equation, adding a constant is the analogous symmetry: translations are isometries of the target $\mathbb{R}$.

Example 1.14 (Mean curvature flow)**.** Let

$$F : M^n \hookrightarrow \mathbb{R}^{n+k}$$

be a smooth immersion of a closed manifold. Define the area functional by

$$\mathcal{A}(F) = \int_M d\mu_F,$$

where $d\mu_F$ is the measure induced by F.

The tangent space to the space of immersions at F can be identified with

$$T_F X = \Gamma(F^* T\mathbb{R}^{n+k}).$$

Equip this space with the L^2 metric

$$G_F(V, W) = \int_M \langle V, W \rangle \, d\mu_F.$$

For a variation F_s with variation field

$$V = \frac{d}{ds}\bigg|_{s=0} F_s,$$

the first variation of area is

$$\frac{d}{ds}\bigg|_{s=0} \mathcal{A}(F_s) = -\int_M \left\langle \vec{H}, V \right\rangle d\mu_F,$$

where $\vec{H}$ is the mean-curvature vector. Therefore

$$\nabla \mathcal{A}(F) = -\vec{H},$$

and the negative gradient flow is

$$\partial_t F = \vec{H}.$$

This is mean curvature flow.

Critical points satisfy

$$\vec{H} = 0$$

and are called **minimal immersions** or **minimal submanifolds**.

Mean curvature flow also has self-similar solutions. In particular, self-shrinkers arise as tangent flows at singularities. They are not critical points of ordinary area; after rescaling the flow, they become critical points of a Gaussian weighted area functional.

1.4 Linearization and Morse–Bott critical points

The next goal is to understand how the Hessian of the energy controls the behavior of a gradient flow near a critical point.

Let

$$f : \mathbb{R}^n \to \mathbb{R}$$

be smooth, and suppose

$$\nabla f(0) = 0.$$

Set

$$A = D(\nabla f)_0 = \operatorname{Hess} f(0).$$

After identifying the Hessian with a symmetric matrix, Taylor expansion gives

$$f(x) = f(0) + \frac{1}{2} \langle Ax, x \rangle + O(|x|^3)$$

and

$$\nabla f(x) = Ax + O(|x|^2).$$

Thus A is the linearization of the generally nonlinear map $x \mapsto \nabla f(x)$.

The nonlinear gradient-flow equation is

$$x' = -\nabla f(x).$$

Its linearization at the critical point is

$$\xi' = -A\xi.$$

Suppose first that A is positive definite. Then there is a $\lambda > 0$ such that

$$\langle A\xi, \xi \rangle \geq \lambda |\xi|^2$$

for every $\xi \in \mathbb{R}^n$. Hence

$$\frac{d}{dt} \frac{1}{2} |\xi|^2 = -\langle A\xi, \xi \rangle \leq -\lambda |\xi|^2 .$$

It follows that

$$|\xi(t)| \leq e^{-\lambda t} |\xi(0)| .$$

Thus positive eigenvalues of the Hessian produce exponentially decaying directions. Negative eigenvalues produce unstable directions, while zero eigenvalues are not resolved by the linearized equation.

Definition 1.15. A critical point p of f is **Morse** if

$$\operatorname{Hess} f(p)$$

is nondegenerate.

The **critical set** of f is

$$\mathcal{C} = \{p : \nabla f(p) = 0\}.$$

A Morse critical point is isolated. We now consider the case in which the Hessian is degenerate:

$$\ker \operatorname{Hess} f(p) \neq \{0\}.$$

There are two qualitatively different possibilities.

1. The kernel consists precisely of directions tangent to a smooth family of critical points.

   For example, let
   $$f(x,y) = \frac{1}{2}x^2.$$
   Then
   $$\mathcal{C} = \{(0,y) : y \in \mathbb{R}\}.$$
   For each $p \in \mathcal{C}$,
   $$\ker \operatorname{Hess} f(p) = \mathbb{R} \cdot (0,1) = T_p\mathcal{C}.$$
   Here $\mathbb{R} \cdot (0,1)$ denotes the vector space spanned by $(0,1)$. The Hessian is degenerate only in the direction along which one moves through other critical points. It remains nondegenerate transverse to the critical set.

2. The kernel contains directions that do not arise from a smooth family of critical points.

   For example, let
   $$f(x,y) = \frac{1}{2}x^2 + \frac{1}{4}y^4.$$
   Then
   $$\mathcal{C} = \{(0,0)\}.$$
   At $p = (0,0)$,
   $$\ker \operatorname{Hess} f(p) = \mathbb{R} \cdot (0,1),$$
   whereas
   $$T_p\mathcal{C} = \{0\}.$$
   The Hessian cannot detect the leading y^4-term, so the y-direction is a genuinely higher-order degeneracy.

Definition 1.16. A critical point $p \in \mathcal{C}$ is **Morse–Bott** if, in a neighborhood of p,

- the critical set $\mathcal{C}$ is a smooth submanifold, and
- $$T_p\mathcal{C} = \ker \operatorname{Hess} f(p).$$

The function f is called **Morse–Bott** if every critical point is Morse–Bott.

The Morse–Bott lemma gives local coordinates (x,y) centered at p such that
$$\mathcal{C} = \{y = 0\}$$
and
$$f(x,y) = f(p) + \frac{1}{2}\langle Ay, y\rangle,$$
where A is nondegenerate. More explicitly, after another linear change of coordinates, the quadratic term can be written as a sum of positive and negative squares.

Consequently, for (x,y) sufficiently close to p,
$$|f(x,y) - f(p)| \leq C\,|y|^2$$

and
$$|\nabla f(x,y)| \geq c\,|y|\,.$$
It follows that
$$|f(x,y) - f(p)|^{1/2} \leq C'\,|\nabla f(x,y)|\,.$$
This is a Łojasiewicz gradient inequality with the optimal exponent $\theta = \frac{1}{2}$. As seen above, such an inequality yields finite length of nearby gradient-flow trajectories.

Two important sources of degeneracy are:

- symmetries of the energy functional;
- a finite-dimensional kernel of the Hessian.

When the Hessian is Fredholm, its kernel is finite-dimensional. One can often reduce the problem to this kernel using Lyapunov–Schmidt reduction.

2 Degeneracies

2.1 Symmetries

Let X be a suitable manifold, let
$$\mathcal{E} : X \to \mathbb{R}, \qquad \mathcal{M} = \nabla\mathcal{E},$$
and let u_* be a critical point:
$$\mathcal{M}(u_*) = 0.$$
The linearization of the gradient at u_* is
$$L = D\mathcal{M}_{u_*}.$$
After identifying tangent and cotangent spaces using the metric on X, the operator L represents the Hessian of $\mathcal{E}$ at u_*.

Suppose that a Lie group G acts smoothly on X and that the energy is invariant:
$$\mathcal{E}(g \cdot u) = \mathcal{E}(u) \qquad \text{for every } g \in G,\ u \in X.$$
If u_* is critical, then every point of its orbit
$$G \cdot u_* = \{g \cdot u_* : g \in G\}$$
is also critical.

Let
$$\mathrm{Lie}(G) = T_eG$$
denote the Lie algebra of G. For $\xi \in \mathrm{Lie}(G)$, define the induced infinitesimal action by
$$\xi_X(u) = \left.\frac{d}{ds}\right|_{s=0} \exp(s\xi) \cdot u.$$
Since
$$\mathcal{M}(\exp(s\xi) \cdot u_*) = 0$$

for every s, differentiation at $s = 0$ gives

$$L\big(\xi_X(u_*)\big) = 0.$$

Thus infinitesimal symmetry directions lie in $\ker L$. Symmetries therefore produce degeneracy of the Hessian.

Definition 2.1. A **slice** through u_* is a submanifold $S \subset X$ containing u_* that is transverse to the orbit $G \cdot u_*$ and captures nearby points modulo the action of G.

A strong local slice condition is that the map

$$G \times S \to X, \qquad (g, u) \mapsto g \cdot u,$$

is locally surjective near (e, u_*).

The corresponding infinitesimal condition is

$$T_{u_*}X = T_{u_*}(G \cdot u_*) \oplus T_{u_*}S.$$

The infinitesimal direct-sum condition is weaker by itself; additional analytic work is generally needed to obtain an actual local slice.

Restricting the problem to a slice removes the degeneracy caused by the group orbit, at least locally. Other kernel directions may still remain.

2.2 Fredholm linearizations

In PDE applications, we would like the Hessian operator L to be Fredholm. One must first choose appropriate Banach or Hilbert spaces so that

$$L : \mathcal{X} \to \mathcal{Y}$$

is a bounded linear operator. Alternatively, one can formulate the theory for suitable unbounded operators on a Hilbert space.

Remark 2.2. In applications, the same differential operator often has two compatible realizations. It is a bounded Fredholm operator

$$L : \mathcal{X} \to \mathcal{Y}$$

when the domain is equipped with a stronger Banach norm, and it is an unbounded self-adjoint operator on a Hilbert space H with domain $\mathcal{X} \subset H$.

Definition 2.3. A bounded linear operator

$$L : \mathcal{X} \to \mathcal{Y}$$

between Banach spaces is **Fredholm** if

1. $\ker L$ is finite-dimensional;
2. $\operatorname{im} L$ is closed;
3. the cokernel

   $$\operatorname{coker} L = \mathcal{Y}/\operatorname{im} L$$

   is finite-dimensional.

The **index** of L is

$$\operatorname{ind}(L) = \dim \ker L - \dim \operatorname{coker} L.$$

If H is a Hilbert space and

$$L : H \to H$$

is bounded, self-adjoint, and Fredholm, then

$$(\operatorname{im} L)^{\perp} = \ker L.$$

Since $\operatorname{im} L$ is closed, it follows that

$$H = \ker L \oplus (\ker L)^{\perp} = \ker L \oplus \operatorname{im} L.$$

In particular, a self-adjoint Fredholm operator has index zero.

Fredholmness does not by itself imply that the critical point is Morse–Bott. It does, however, imply that all degeneracy is confined to the finite-dimensional space $\ker L$.

The Lyapunov–Schmidt procedure uses the decomposition

$$H = \ker L \oplus \operatorname{im} L$$

to solve the critical-point equation in the infinite-dimensional $\operatorname{im} L$ direction. This leaves a finite-dimensional reduced function defined on $\ker L$. One can then apply the classical finite-dimensional Łojasiewicz inequality to the reduced function, provided the relevant maps are analytic.

In the Morse–Bott case, the reduced function is locally constant. In a more degenerate situation, it may contain nontrivial higher-order terms, such as the y^4 term in

$$f(x, y) = \frac{1}{2}x^2 + \frac{1}{4}y^4.$$

Now we consider the finite-dimensional Łojasiewicz inequalities in more detail. We will then prove a local Łojasiewicz–Simon inequality for analytic functionals whose Hessians are Fredholm. Finally, we will apply this inequality to harmonic map heat flow.

3 Finite-dimensional Łojasiewicz inequalities

Theorem 3.1 (Łojasiewicz, 1962)**.** *Let $U \subset \mathbb{R}^n$ be open, let $f : U \to \mathbb{R}$ be real analytic, and let $p \in U$ satisfy*

$$\nabla f(p) = 0.$$

Then there exist constants

$$C > 0, \qquad \sigma > 0, \qquad \theta \in \left(0, \frac{1}{2}\right],$$

such that

$$|f(x) - f(p)|^{1-\theta} \le C\,|\nabla f(x)| \tag{7}$$

whenever

$$|x - p| < \sigma.$$

After replacing f by $f - f(p)$, assume that $f(p) = 0$ and define

$$Z = \{x \in U : f(x) = 0\}.$$

Then there also exist constants

$$C > 0, \qquad \sigma > 0, \qquad \alpha \in [1, \infty)$$

such that

$$\operatorname{dist}(x, Z)^\alpha \le C\,|f(x)| \tag{8}$$

whenever

$$|x - p| < \sigma.$$

Remark 3.2. The first inequality is called the **Łojasiewicz gradient inequality**. The second is a quantitative comparison between the vanishing of an analytic function and the distance to its zero set.

The exponent θ is generally not unique: if the inequality holds for one exponent, it may also hold for smaller exponents after shrinking the neighborhood. The largest admissible exponent gives the strongest inequality and the best convergence rate.

Example 3.3. Consider

$$f(x, y) = \frac{1}{2}x^2 + \frac{1}{4}y^4.$$

The origin is an isolated critical point, but it is not Morse–Bott, since

$$\ker \operatorname{Hess} f(0, 0) = \mathbb{R} \cdot (0, 1)$$

while the tangent space to the critical set is zero.

Nevertheless, f is analytic, so the Łojasiewicz gradient inequality holds. The optimal exponent is

$$\theta = \frac{1}{4}.$$

Indeed, along the y-axis,

$$f(0, y) = \frac{1}{4}y^4, \qquad |\nabla f(0, y)| = |y|^3.$$

Thus an inequality

$$|f|^{1-\theta} \le C\,|\nabla f|$$

requires

$$|y|^{4(1-\theta)} \lesssim |y|^3,$$

which is possible near zero only when

$$\theta \le \frac{1}{4}.$$

By contrast, a Morse–Bott critical point has the optimal exponent $\theta = \frac{1}{2}$.

We are often interested in measuring how far a point is from the critical set of a function rather than from one of its level sets.

Corollary 3.4. Let $f : U \to \mathbb{R}$ be real analytic, let p be a critical point, and let

$$\mathcal{C} = \{x \in U : \nabla f(x) = 0\}.$$

Then there exist constants

$$C > 0, \qquad \sigma > 0, \qquad \alpha \in [1, \infty)$$

such that

$$\operatorname{dist}(x, \mathcal{C})^{\alpha} \le C\, |\nabla f(x)|^2$$

whenever

$$|x - p| < \sigma.$$

Proof. Apply the zero-set inequality to the analytic function

$$g(x) = |\nabla f(x)|^2 .$$

Its zero set is exactly the critical set of f:

$$g^{-1}(0) = \mathcal{C}.$$

□

Remark 3.5. Real analyticity, or some related tameness assumption, is essential. The theorem is false for arbitrary smooth functions.

Exercise 3.6. Let

$$f(x) = \begin{cases} e^{-x^{-2}}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Show that f is smooth but not real analytic at zero, and that for every

$$\theta \in \left(0, \frac{1}{2}\right]$$

one has

$$\frac{f(x)^{1-\theta}}{|f'(x)|} \longrightarrow \infty \qquad \text{as } x \downarrow 0.$$

Solution. For $x > 0$,

$$f'(x) = 2x^{-3} e^{-x^{-2}}.$$

Therefore

$$\begin{aligned} \frac{f(x)^{1-\theta}}{|f'(x)|} &= \frac{e^{-(1-\theta)x^{-2}}}{2x^{-3}e^{-x^{-2}}} \\ &= \frac{1}{2} x^3 e^{\theta x^{-2}}. \end{aligned}$$

Since exponential growth dominates polynomial decay,

$$\frac{1}{2} x^3 e^{\theta x^{-2}} \longrightarrow \infty$$

for every $\theta > 0$. □

4 The infinite-dimensional Łojasiewicz–Simon inequality

Let H be a Hilbert space and let X be a Banach space continuously and densely embedded in H:

$$X \hookrightarrow H.$$

A typical example is

$$X = H^2(M), \qquad H = L^2(M),$$

where M is a closed manifold.

Definition 4.1. Let $U \subset X$ be open. A map

$$F : U \to H$$

is **real analytic** at $u_0 \in U$ if there exist continuous symmetric k-linear maps

$$A_k : X^k \to H, \qquad k \geq 1,$$

and $r > 0$ such that

$$F(u_0 + h) = F(u_0) + \sum_{k=1}^{\infty} A_k(h, \ldots, h)$$

for every $||h||_X < r$, with

$$\sum_{k=1}^{\infty} ||A_k|| \, ||h||_X^k < \infty.$$

In this convention,

$$A_k = \frac{1}{k!} D^k F(u_0),$$

or equivalently

$$D^k F(u_0) = k! A_k.$$

Let

$$\mathcal{E} : U \subset X \to \mathbb{R}$$

be an energy functional. Suppose that its derivative is represented using the inner product of H by an Euler–Lagrange operator

$$\mathcal{M} : U \to H,$$

meaning that

$$D\mathcal{E}_u(v) = \langle \mathcal{M}(u), v \rangle_H$$

for every $u \in U$ and $v \in X$.

Theorem 4.2 (Łojasiewicz–Simon inequality)**.** *Suppose that*

- $\mathcal{E} : U \subset X \to \mathbb{R}$ *is real analytic;*
- $\mathcal{M} : U \to H$ *is its analytic Euler–Lagrange operator;*
- $u_* \in U$ *is a critical point:*

$$\mathcal{M}(u_*) = 0;$$

- *the linearized operator*

$$L = D\mathcal{M}(u_*) : X \to H$$

is Fredholm;

- *L, considered as an operator with domain $X \subset H$, is self-adjoint.*

Then there exist constants

$$C > 0, \qquad \sigma > 0, \qquad \theta \in \left(0, \frac{1}{2}\right]$$

such that

$$|\mathcal{E}(u) - \mathcal{E}(u_*)|^{1-\theta} \leq C \, ||\mathcal{M}(u)||_H$$

whenever

$$||u - u_*||_X < \sigma.$$

Sketch of proof. After translating the critical point and subtracting a constant from the energy, assume

$$u_* = 0, \qquad \mathcal{E}(0) = 0, \qquad \mathcal{M}(0) = 0.$$

Lyapunov–Schmidt reduction. Let

$$K = \ker L.$$

Since L is Fredholm, K is finite-dimensional. Since L is self-adjoint and Fredholm,

$$H = K \oplus K^{\perp}$$

and

$$\operatorname{im} L = K^{\perp}.$$

Let

$$P : H \to K, \qquad Q = I - P : H \to K^{\perp}$$

be the orthogonal projections.

Choose a compatible splitting of the domain,

$$X = K \oplus X_1, \qquad X_1 = X \cap K^{\perp}.$$

The restricted operator

$$L : X_1 \to K^{\perp}$$

is an isomorphism.

Write a nearby point as

$$u = k + w, \qquad k \in K, \quad w \in X_1.$$

Consider the projected equation

$$Q\mathcal{M}(k + w) = 0.$$

Its derivative with respect to w at $(0, 0)$ is

$$D_w(Q\mathcal{M})(0, 0) = L : X_1 \to K^{\perp},$$

which is invertible. The analytic implicit function theorem therefore gives an analytic map

$$\psi : B_\delta(0) \subset K \to X_1$$

such that

$$Q\mathcal{M}(k + \psi(k)) = 0.$$

Define the finite-dimensional Lyapunov–Schmidt graph

$$\Gamma = \{k + \psi(k) : k \in B_\delta(0) \subset K\}.$$

At every point $\bar{u} \in \Gamma$,

$$\mathcal{M}(\bar{u}) \in K.$$

Define the reduced energy

$$f(k) = \mathcal{E}(k + \psi(k)).$$

Since $\mathcal{E}$ and ψ are analytic, f is an analytic function on the finite-dimensional space K.

For $v \in K$,

$$\begin{aligned} Df_k(v) &= D\mathcal{E}_{k+\psi(k)}\big(v + D\psi_k(v)\big) \\ &= \langle \mathcal{M}(k + \psi(k)), v + D\psi_k(v) \rangle_H . \end{aligned}$$

Now

$$\mathcal{M}(k + \psi(k)) \in K, \qquad D\psi_k(v) \in K^\perp,$$

so

$$Df_k(v) = \langle \mathcal{M}(k + \psi(k)), v \rangle_H .$$

Thus, under the induced inner product on K,

$$\nabla f(k) = \mathcal{M}(k + \psi(k)).$$

The finite-dimensional Łojasiewicz inequality therefore gives

$$|\mathcal{E}(k + \psi(k))|^{1-\theta} \leq C \, ||\mathcal{M}(k + \psi(k))||_H .$$

It remains to lift this estimate from Γ to a full neighborhood of the origin.

Let

$$u = k + w$$

and define

$$\bar{u} = k + \psi(k) \in \Gamma, \qquad \eta = u - \bar{u} = w - \psi(k) \in X_1.$$

Since

$$Q\mathcal{M}(\bar{u}) = 0,$$

Taylor expansion gives

$$Q\mathcal{M}(u) = L\eta + R(u, \bar{u}),$$

where, after shrinking the neighborhood,

$$||R(u, \bar{u})||_H \leq \varepsilon \, ||\eta||_X .$$

Using the invertibility of

$$L : X_1 \to K^\perp,$$

we obtain

$$||\eta||_X \le C\,||Q\mathcal{M}(u)||_H \le C\,||\mathcal{M}(u)||_H\,.$$

Moreover,

$$\begin{aligned}\mathcal{E}(u)-\mathcal{E}(\bar{u}) &= \int_0^1 D\mathcal{E}_{\bar{u}+t\eta}(\eta)\,dt\\ &= \int_0^1 \langle \mathcal{M}(\bar{u}+t\eta),\eta\rangle_H\,dt.\end{aligned}$$

Since

$$\mathcal{M}(\bar{u}) \in K, \qquad \eta \in K^{\perp},$$

the zeroth-order term vanishes:

$$\langle \mathcal{M}(\bar{u}),\eta\rangle_H = 0.$$

Using the local Lipschitz continuity of $\mathcal{M}$,

$$|\mathcal{E}(u)-\mathcal{E}(\bar{u})| \le C\,||\eta||_X^2 \le C\,||\mathcal{M}(u)||_H^2\,.$$

One also obtains

$$||\mathcal{M}(\bar{u})||_H \le C\,||\mathcal{M}(u)||_H\,.$$

Combining these estimates with the finite-dimensional inequality on Γ yields

$$|\mathcal{E}(u)|^{1-\theta} \le C\,||\mathcal{M}(u)||_H\,.$$

□

4.1 Integrable Jacobi fields

Definition 4.3. Let

$$K = \ker L.$$

An element $v \in K$ is called a **Jacobi field**. It is **integrable** if there exists a C^1 curve

$$\gamma : (-\epsilon,\epsilon) \to X$$

such that

$$\gamma(0) = u_*, \qquad \gamma'(0) = v, \qquad \mathcal{M}(\gamma(s)) = 0$$

for every $s \in (-\epsilon,\epsilon)$.

The critical point u_* is called **fully integrable** if the critical set is, in a neighborhood of u_*, a smooth finite-dimensional submanifold $\mathcal{C}_*$ satisfying

$$u_* \in \mathcal{C}_*, \qquad T_{u_*}\mathcal{C}_* = K.$$

Equivalently, every kernel direction belongs to one smooth local family of critical points, with no additional nearby critical branches.

Every tangent vector to a smooth family of critical points lies in $\ker L$. Indeed, differentiating

$$\mathcal{M}(\gamma(s)) = 0$$

at $s = 0$ gives

$$L\gamma'(0) = 0.$$

Full integrability asserts the reverse inclusion and identifies the nearby critical set as a smooth manifold tangent to K. This is the infinite-dimensional Morse–Bott situation.

Theorem 4.4. *Suppose that $u_* = 0$ is a fully integrable critical point and that the standard analytic and Fredholm hypotheses above hold. Then there exist constants*

$$C > 0, \qquad \sigma > 0$$

such that

$$|\mathcal{E}(u) - \mathcal{E}(0)|^{1/2} \leq C \, ||\mathcal{M}(u)||_H$$

whenever

$$||u||_X < \sigma.$$

Idea of proof. Perform the Lyapunov–Schmidt reduction. Full integrability implies that the Lyapunov–Schmidt graph

$$\Gamma = \{k + \psi(k) : k \in K\}$$

is locally the critical set. Consequently, the reduced energy

$$f(k) = \mathcal{E}(k + \psi(k))$$

is locally constant.

Thus there is no higher-order variation of the energy in the kernel directions. All variation occurs in the directions transverse to the critical manifold, where the Hessian is nondegenerate. The same quadratic estimate as in the Morse–Bott lemma then gives the exponent

$$\theta = \frac{1}{2}.$$

□

5 Applications to harmonic map heat flow

5.1 Convergence near a smooth harmonic map

Let (Σ^2, g) be a closed surface and let (N, h) be a closed real-analytic Riemannian manifold. Let

$$\phi : \Sigma \to N$$

be a smooth harmonic map:

$$\tau(\phi) = 0.$$

Set

$$X = H^2(\Sigma, \phi^* TN), \qquad H = L^2(\Sigma, \phi^* TN).$$

Since $\dim \Sigma = 2$, the Sobolev embedding

$$H^2(\Sigma) \hookrightarrow C^0(\Sigma)$$

implies that a sufficiently small section $\xi \in X$ lies pointwise in a normal neighborhood of the zero section.

Define

$$\mathrm{Exp}_\phi(\xi)(x) = \exp_{\phi(x)} \xi(x).$$

This gives a coordinate chart from a neighborhood of zero in $H^2(\Sigma, \phi^*TN)$ to a neighborhood of ϕ in the space of maps $\Sigma \to N$.

Define the coordinate representation of the energy by

$$\widetilde{\mathcal{E}}(\xi) = \mathcal{E}(\mathrm{Exp}_\phi(\xi)).$$

The tension field

$$\tau(\mathrm{Exp}_\phi(\xi))$$

is naturally a section of the varying bundle

$$\mathrm{Exp}_\phi(\xi)^*TN.$$

Let

$$\mathcal{P}_\xi : \mathrm{Exp}_\phi(\xi)^*TN \to \phi^*TN$$

denote parallel transport along the geodesics

$$s \longmapsto \exp_{\phi(x)}(s\xi(x)),$$

and set

$$\mathcal{A}_\xi = \mathcal{P}_\xi \circ D(\mathrm{Exp}_\phi)_\xi.$$

The Euler–Lagrange operator in these fixed coordinates is

$$\widetilde{\mathcal{M}}(\xi) = -\mathcal{A}_\xi^* \mathcal{P}_\xi\big(\tau(\mathrm{Exp}_\phi(\xi))\big).$$

It is characterized by

$$D\widetilde{\mathcal{E}}_\xi(\eta) = \left\langle \widetilde{\mathcal{M}}(\xi), \eta \right\rangle_{L^2}.$$

Since $\tau(\phi) = 0$ and $\mathcal{A}_0 = I$, the linearization at zero is, up to sign conventions, the Jacobi operator:

$$D\widetilde{\mathcal{M}}(0)\xi = J_\phi \xi.$$

Here, up to sign conventions,

$$J_\phi \xi = \nabla^* \nabla \xi - \mathrm{tr}_g\big(R^N(\xi, d\phi)d\phi\big).$$

The operator

$$J_\phi : H^2(\Sigma, \phi^*TN) \to L^2(\Sigma, \phi^*TN)$$

is elliptic and self-adjoint. Since Σ is closed, it is Fredholm.

The analyticity of (N, h) implies that, in these local coordinates, the energy and its fixed-coordinate Euler–Lagrange operator are analytic. Hence the Łojasiewicz–Simon inequality applies near ϕ.

Fix an isometric embedding

$$N \hookrightarrow \mathbb{R}^L.$$

Unless otherwise stated, Sobolev and Hölder norms of maps into N are understood extrinsically using this embedding. Near a fixed map ϕ, these norms are equivalent to the corresponding norms in exponential-map coordinates.

Theorem 5.1 (Simon, 1983)**.** *Suppose that (N, h) is closed and real analytic. Let*

$$u : \Sigma \times [0, \infty) \to N$$

be a smooth solution of harmonic map heat flow

$$\partial_t u = \tau(u)$$

defined for all time.

Suppose there exist times

$$t_j \to \infty$$

and a smooth harmonic map

$$u_\infty : \Sigma \to N$$

such that

$$u(t_j) \to u_\infty \qquad \text{in } C^2.$$

Then

$$u(t) \to u_\infty \qquad \text{in } C^2$$

as $t \to \infty$.

Parabolic bootstrapping then gives convergence in C^∞.

Remark 5.2. Eells–Sampson theory gives global smooth existence, for example, when the target has nonpositive sectional curvature.

Sketch of proof. Fix a small number $\delta > 0$ and set

$$s_j = t_j + \delta.$$

Because

$$u(t_j) \to u_\infty \quad \text{in } C^2$$

and the harmonic map heat equation is parabolic, interior parabolic regularity implies that, for every $m \geq 0$,

$$u(s_j) \to u_\infty \quad \text{in } C^m.$$

In particular,

$$u(s_j) \to u_\infty \quad \text{in } H^2.$$

The positive time shift δ is fixed. Parabolic smoothing estimates at time $t_j + \delta$ depend on δ and on uniform C^2 bounds for the initial data at time t_j. Such uniform bounds hold for all sufficiently large j because $u(t_j) \to u_\infty$ in C^2.

Choose $\sigma > 0$ sufficiently small that the Łojasiewicz–Simon inequality holds in the H^2-ball

$$B_{H^2}(u_\infty, \sigma).$$

We claim that, for all sufficiently large j, the trajectory remains in this ball for every $t \geq s_j$.

Suppose otherwise. Let $T_* > s_j$ be the first time at which

$$||u(T_*) - u_\infty||_{H^2} = \sigma.$$

Then

$$||u(t) - u_\infty||_{H^2} < \sigma$$

for all

$$s_j \leq t < T_*.$$

Since

$$\mathcal{E}(u(t))$$

is nonincreasing and

$$\mathcal{E}(u(t_j)) \to \mathcal{E}(u_\infty),$$

we have

$$\mathcal{E}(u(t)) \geq \mathcal{E}(u_\infty)$$

for every finite t. Define

$$e(t) = \mathcal{E}(u(t)) - \mathcal{E}(u_\infty).$$

Along harmonic map heat flow,

$$e'(t) = -\left|\left|\tau(u(t))\right|\right|_{L^2}^2.$$

The Łojasiewicz–Simon inequality gives

$$e(t)^{1-\theta} \leq C \left|\left|\tau(u(t))\right|\right|_{L^2}$$

for $s_j \leq t < T_*$. Consequently,

$$\begin{aligned}
-\frac{d}{dt} e(t)^\theta &= \theta e(t)^{\theta-1} \left|\left|\tau(u(t))\right|\right|_{L^2}^2 \\
&\geq c \left|\left|\tau(u(t))\right|\right|_{L^2}.
\end{aligned}$$

Since

$$\partial_t u = \tau(u),$$

integration gives

$$\int_{s_j}^{t} ||\partial_s u(s)||_{L^2} \, ds \leq C e(s_j)^\theta$$

for every $s_j \leq t < T_*$.

Therefore

$$||u(t) - u(s_j)||_{L^2} \leq C e(s_j)^\theta.$$

Because

$$u(s_j) \to u_\infty \quad \text{and} \quad e(s_j) \to 0,$$

we obtain

$$\sup_{s_j \leq t < T_*} ||u(t) - u_\infty||_{L^2} \longrightarrow 0.$$

As long as the trajectory remains in the fixed H^2-neighborhood, parabolic regularity gives uniform higher-order bounds. For sufficiently large $m > 2$,

$$\sup_{s_j \le t < T_*} ||u(t) - u_\infty||_{H^m} \le C_m.$$

Interpolation gives

$$||v||_{H^2} \le C\, ||v||_{L^2}^{1-2/m}\, ||v||_{H^m}^{2/m}.$$

Applying this to

$$v = u(t) - u_\infty$$

shows that

$$\sup_{s_j \le t < T_*} ||u(t) - u_\infty||_{H^2} \longrightarrow 0.$$

For sufficiently large j, this is strictly smaller than σ, contradicting the definition of T_*.

Thus the flow never leaves the Łojasiewicz–Simon neighborhood after time s_j.

The finite-length estimate now gives

$$\int_{s_j}^{\infty} ||\partial_t u(t)||_{L^2}\, dt < \infty,$$

so $u(t)$ converges in L^2 to a limit. Since the subsequence $u(t_j)$ converges to u_∞, that limit must be u_∞. Uniform parabolic estimates and interpolation then improve the convergence to C^2, and parabolic bootstrapping gives convergence in C^∞. □

5.2 Bubbling and loss of compactness

One important application of the Łojasiewicz–Simon inequality is the following principle:

> If a gradient flow has a subsequence converging strongly to a smooth critical point, then a local Łojasiewicz–Simon inequality near that critical point can often be used to prove convergence of the entire trajectory.

A difficulty arises when the subsequential limit is not represented by one smooth map in a strong topology. In dimension two, energy can concentrate at isolated points, producing harmonic spheres called **bubbles**. The limiting object may then be a bubble tree rather than a single smooth map.

Along harmonic map heat flow,

$$\frac{d}{dt}\mathcal{E}(u(t)) = -\, ||\tau(u(t))||_{L^2}^2.$$

Therefore

$$\int_0^{\infty} ||\tau(u(t))||_{L^2}^2\, dt < \infty,$$

and there exists a sequence $t_j \to \infty$ such that

$$||\tau(u(t_j))||_{L^2} \to 0.$$

Thus the sequence

$$u_j = u(t_j)$$

is a sequence of almost-harmonic maps.

5.2.1 Bubbles and bubble trees

Definition 5.3. Let

$$u_j : \Sigma \to N$$

be a sequence satisfying

$$\mathcal{E}(u_j) \leq M, \qquad ||\tau(u_j)||_{L^2} \to 0.$$

Suppose there are points

$$x_j \to x \in \Sigma$$

and scales

$$r_j \downarrow 0$$

such that the rescaled maps

$$v_j(y) = u_j\big(\exp_{x_j}(r_j y)\big)$$

converge on compact subsets of $\mathbb{R}^2$, away possibly from finitely many additional concentration points, to a nonconstant finite-energy harmonic map

$$\omega : \mathbb{R}^2 \to N.$$

Then ω is called a **bubble**.

To see why the limit is harmonic, consider first flat coordinates, in which

$$v_j(y) = u_j(x_j + r_j y).$$

The tension scales according to

$$\tau(v_j)(y) = r_j^2 \tau(u_j)(x_j + r_j y).$$

Hence, for every fixed $R > 0$,

$$\begin{aligned} ||\tau(v_j)||^2_{L^2(B_R)} &= r_j^2 \int_{B_{r_j R}(x_j)} |\tau(u_j)|^2 \\ &\leq r_j^2 \, ||\tau(u_j)||^2_{L^2(\Sigma)} \longrightarrow 0. \end{aligned}$$

The same conclusion holds in exponential coordinates because the rescaled domain metrics converge smoothly to the Euclidean metric.

A finite-energy harmonic map

$$\omega : \mathbb{R}^2 \to N$$

extends smoothly across the point at infinity. Thus it can be regarded as a harmonic sphere

$$\omega : S^2 \to N.$$

At the exceptional points in the convergence of v_j, further bubbles may form at still smaller scales. Repeating this procedure gives a **bubble tree**. The original weak limit is called the **body map**, and the harmonic spheres arising at the various scales are the bubble maps.

The extraction process terminates after finitely many steps. Indeed, there is a constant $\varepsilon_N > 0$ such that every nonconstant harmonic sphere has energy at least ε_N. Since the total energy is bounded by M, only finitely many nonconstant bubbles can occur.

Remark 5.4. The standard local Łojasiewicz–Simon inequality near a smooth harmonic map does not directly apply to a bubbling sequence. Near a bubble of scale r_j, one typically has

$$|du_j| \sim r_j^{-1}, \qquad \left|\nabla^2 u_j\right| \sim r_j^{-2}.$$

Thus the maps need not remain bounded, let alone close to the body map, in H^2. A bubble tree is a singular limiting configuration involving collapsing scale parameters, rather than a single point of the ordinary H^2 configuration space.

5.2.2 Strong regularity rules out bubbling

Suppose $\dim \Sigma = 2$ and

$$\sup_j ||u_j||_{H^2(\Sigma)} \leq C.$$

Then

$$du_j \in H^1(\Sigma),$$

and the two-dimensional Sobolev embedding gives

$$\sup_j ||du_j||_{L^p(\Sigma)} \leq C_p$$

for every finite p.

Fix $p > 2$. Hölder's inequality gives

$$\begin{aligned}\mathcal{E}(u_j; B_r(x)) &= \frac{1}{2}\int_{B_r(x)} |du_j|^2 \\ &\leq \frac{1}{2}\,||du_j||^2_{L^p(B_r(x))} \operatorname{Vol}(B_r(x))^{1-\frac{2}{p}} \\ &\leq C r^{2(1-\frac{2}{p})}.\end{aligned}$$

Since $p > 2$,

$$2\left(1 - \frac{2}{p}\right) > 0,$$

and therefore

$$\lim_{r\downarrow 0} \sup_j \sup_{x\in\Sigma} \mathcal{E}(u_j; B_r(x)) = 0.$$

Thus no energy can concentrate, and bubbling is impossible.

In dimension n, the scale-invariant energy is

$$r^{2-n}\int_{B_r} |du|^2 .$$

A uniform $W^{1,p}$ bound gives

$$r^{2-n}\int_{B_r} |du|^2 \leq C r^{2-\frac{2n}{p}}.$$

To force the scale-invariant energy to vanish as $r \to 0$, one needs

$$p > n.$$

Thus in higher dimensions one must work in a topology strong enough to control du in some L^p space with $p > n$.

5.2.3 Energy quantization for maps between two-spheres

Every nonconstant harmonic map

$$\omega : S^2 \to S^2$$

is holomorphic or antiholomorphic, and

$$\mathcal{E}(\omega) = 4\pi \, |\deg \omega| \, .$$

Consequently, the total energy of a bubble tree of maps $S^2 \to S^2$ is an integer multiple of 4π:

$$\mathcal{E}_{\text{tree}} = 4\pi k$$

for some $k \in \mathbb{Z}_{\geq 0}$.

The integer k is determined by the degrees of the body and bubble maps; it is not an index in the Fredholm-theoretic sense.

Theorem 5.5 (Topping, 2004)**.** *Let*

$$u_j : S^2 \to S^2$$

be smooth maps satisfying

$$\mathcal{E}(u_j) \leq M, \qquad ||\tau(u_j)||_{L^2} \to 0.$$

After passing to a subsequence, suppose the body map u_∞ is holomorphic and that, at each bubble point, either

- *all bubbles are holomorphic, or*
- *all bubbles are antiholomorphic and*

  $$|\nabla u_\infty| \neq 0$$

  at that bubble point.

Then there are constants $C > 0$ and $k \in \mathbb{Z}_{\geq 0}$ such that

$$|\mathcal{E}(u_j) - 4\pi k| \leq C \, ||\tau(u_j)||_{L^2}^2 \, .$$

Equivalently,

$$|\mathcal{E}(u_j) - 4\pi k|^{1/2} \leq C \, ||\tau(u_j)||_{L^2} \, .$$

This is an analogue of a Łojasiewicz inequality with the optimal exponent $\theta = \frac{1}{2}$, but it is valid near certain singular bubble trees rather than only near one smooth critical map.

Theorem 5.6 (Waldron, schematic form)**.** *For maps*

$$u : S^2 \to S^2$$

with controlled energy and small tension, there are energy-quantization estimates of the form

$$|\mathcal{E}(u) - 4\pi k| \leq C \, ||\tau(u)||_{L^2}^\alpha \, .$$

An exponent $\alpha = 1$ is available in general, while $\alpha > 1$ is available under an additional nondegeneracy condition.

Writing

$$1-\theta = \frac{1}{\alpha},$$

an estimate with $\alpha > 1$ becomes a gradient inequality

$$|\mathcal{E}(u) - 4\pi k|^{1-\theta} \leq C \, ||\tau(u)||_{L^2}$$

with $\theta > 0$.

Remark 5.7. There are several related results near simple bubble trees.

Malchiodi, Rupflin, and Sharp proved Łojasiewicz inequalities near simple bubble trees for an H-functional on closed non-spherical surfaces.

Rupflin proved analogous inequalities for the harmonic map energy for maps from positive-genus surfaces into analytic target manifolds near a simple bubble tree, under suitable hypotheses on the attachment point of the bubble.

6 Applications to mean curvature flow

Let

$$F : M^n \hookrightarrow \mathbb{R}^{n+1}$$

be a smooth immersion of a closed hypersurface. Choose a unit normal ν and write a normal variation as

$$V = f\nu.$$

Using the convention

$$\vec{H} = -H\nu,$$

the first variation of area is

$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=0} \mathrm{Area}(F_s) &= -\int_M \left\langle \vec{H}, V \right\rangle \, d\mu \\ &= \int_M Hf \, d\mu. \end{aligned}$$

Thus the negative L^2-gradient flow of area is

$$(\partial_t F)^{\perp} = \vec{H}.$$

Tangential components of $\partial_t F$ only change the parametrization.

6.1 Graphical mean curvature flow

Locally, write the evolving hypersurface as a graph

$$F(y,t) = (y, u(y,t)), \qquad y \in \Omega \subset \mathbb{R}^n.$$

Then u satisfies

$$\begin{aligned} \partial_t u &= \sqrt{1+|Du|^2}\, \mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \\ &= \left(\delta^{ij} - \frac{u_i u_j}{1+|Du|^2}\right) u_{ij}. \end{aligned}$$

The coefficient matrix
$$a^{ij}(Du) = \delta^{ij} - \frac{u_i u_j}{1 + |Du|^2}$$
is positive definite. It is uniformly positive definite whenever $|Du|$ is bounded. Thus the graphical equation is quasilinear parabolic and gives local short-time existence.

For a compact smooth mean curvature flow with maximal existence interval $[0, T)$, the flow extends smoothly past T if
$$\sup_{0 \le t < T} \sup_{M_t} |A| < \infty.$$
Consequently, if $T < \infty$ is a maximal singular time, then
$$\limsup_{t \uparrow T} \sup_{M_t} |A| = \infty.$$

Definition 6.1 (Avoidance principle)**.** If two compact mean curvature flows are disjoint at some initial time, then they remain disjoint for as long as both flows are smooth.

As a consequence, every closed embedded hypersurface develops a singularity in finite time. Indeed, enclose the initial hypersurface in a sufficiently large round sphere. The sphere shrinks to a point in finite time, and avoidance forces the original hypersurface to remain inside it.

6.2 Huisken's monotonicity formula

Fix a space-time point
$$(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}.$$
For $t < t_0$, define the backward heat kernel
$$\rho_{x_0,t_0}(x, t) = \frac{1}{(4\pi(t_0 - t))^{n/2}} \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right).$$
Define the Gaussian density ratio
$$\Phi_{x_0,t_0}(t) = \int_{M_t} \rho_{x_0,t_0}(x, t)\, d\mu_t(x).$$

Huisken's monotonicity formula states that
$$\frac{d}{dt}\Phi_{x_0,t_0}(t) = -\int_{M_t} \left|\vec{H} + \frac{(x - x_0)^\perp}{2(t_0 - t)}\right|^2 \rho_{x_0,t_0}\, d\mu_t \le 0. \tag{9}$$

Equality holds on a time interval precisely when
$$\vec{H} = -\frac{(x - x_0)^\perp}{2(t_0 - t)}.$$
This is the self-shrinker equation in unrescaled variables.

The **Gaussian density** of the flow at (x_0, t_0) is
$$\Theta_{x_0,t_0} = \lim_{t \uparrow t_0} \Phi_{x_0,t_0}(t).$$
The limit exists by monotonicity.

6.3 Tangent flows

Let $\lambda_j \to \infty$. Define the parabolically rescaled flows

$$M_s^{(j)} = \lambda_j \left(M_{t_0+\lambda_j^{-2}s} - x_0 \right), \qquad s < 0.$$

Definition 6.2. A **tangent flow** at (x_0, t_0) is a subsequential limit of $M_s^{(j)}$ as $\lambda_j \to \infty$, in an appropriate weak sense such as Brakke-flow convergence.

Under the standard local mass bounds, Brakke compactness gives subsequential tangent flows. Huisken's monotonicity formula implies that every tangent flow is backward self-similar.

A tangent flow need not be compact or smooth and may occur with multiplicity. These possibilities prevent a direct application of the ordinary compact Łojasiewicz–Simon theorem.

6.4 Rescaled mean curvature flow

Set

$$s = -\log(t_0 - t)$$

and

$$y = \frac{x - x_0}{\sqrt{t_0 - t}}.$$

Then $s \to \infty$ as $t \uparrow t_0$, and the rescaled hypersurfaces are

$$\widetilde{M}_s = \frac{M_t - x_0}{\sqrt{t_0 - t}}.$$

A direct computation gives

$$(\partial_s y)^\perp = \widetilde{\vec{H}} + \frac{y^\perp}{2},$$

where $\widetilde{\vec{H}}$ is the mean-curvature vector of $\widetilde{M}_s$.

Thus stationary points of rescaled mean curvature flow satisfy

$$\vec{H} + \frac{x^\perp}{2} = 0.$$

Such hypersurfaces are called **self-shrinkers**. If Σ is a self-shrinker, then

$$M_t = x_0 + \sqrt{t_0 - t}\,\Sigma$$

is a self-similarly shrinking solution of ordinary mean curvature flow.

6.5 The Gaussian area functional

Define

$$\rho(x) = (4\pi)^{-n/2} e^{-|x|^2/4}$$

and

$$F(\Sigma) = \int_\Sigma \rho \, d\mu = (4\pi)^{-n/2} \int_\Sigma e^{-|x|^2/4} \, d\mu.$$

For a normal variation V, the first variation is

$$DF_\Sigma(V) = -\int_\Sigma \left\langle \vec{H} + \frac{x^\perp}{2}, V \right\rangle \rho \, d\mu.$$

Thus, with respect to the weighted L^2 metric,

$$\nabla F = -\left(\vec{H} + \frac{x^\perp}{2} \right).$$

Rescaled mean curvature flow is therefore the negative gradient flow of F:

$$(\partial_s y)^\perp = \vec{H} + \frac{y^\perp}{2}.$$

Along rescaled mean curvature flow,

$$\frac{d}{ds} F(\widetilde{M}_s) = -\int_{\widetilde{M}_s} \left| \vec{H} + \frac{y^\perp}{2} \right|^2 \rho \, d\mu.$$

6.6 Łojasiewicz–Simon inequalities near compact shrinkers

Let

$$\Gamma \subset \mathbb{R}^{n+1}$$

be a closed, smooth, embedded self-shrinker. A nearby hypersurface can be represented as a normal graph

$$\Gamma_u = \{p + u(p)\nu(p) : p \in \Gamma\}.$$

The Gaussian area is an analytic functional of u, and its linearized Euler–Lagrange operator is a self-adjoint elliptic operator on the closed manifold Γ. Hence its Hessian is Fredholm. Fix $\alpha \in (0, 1)$; the following is a Hölder-space version of the Łojasiewicz–Simon inequality.

There exist constants

$$C > 0, \qquad \epsilon > 0, \qquad \theta \in \left(0, \frac{1}{2}\right]$$

such that, whenever

$$||u||_{C^{2,\alpha}} < \epsilon,$$

one has

$$|F(\Gamma_u) - F(\Gamma)|^{1-\theta} \leq C \left|\left| \vec{H}_{\Gamma_u} + \frac{x^\perp}{2} \right|\right|_{L^2_\rho(\Gamma_u)},$$

where

$$||V||^2_{L^2_\rho(\Gamma_u)} = \int_{\Gamma_u} |V|^2 \rho \, d\mu.$$

Theorem 6.3 (Schulze, 2014)**.** *Suppose that one tangent flow at a space-time point consists of a closed, multiplicity-one, smoothly embedded self-shrinker Γ. Then that tangent flow is unique: every sequence of parabolic rescalings converges to the same self-shrinking tangent flow.*

The proof combines the local Łojasiewicz–Simon inequality for the Gaussian area with regularity and trapping arguments for the rescaled flow. The compactness of Γ is essential for applying the standard Fredholm and Łojasiewicz–Simon framework directly. Noncompact self-shrinkers require weighted and localized versions of the theory.

Remark 6.4. For a rescaled hypersurface Σ, define the **shrinker defect**

$$S := \vec{H} + \frac{x^\perp}{2}.$$

This quantity measures how far Σ is from being a self-shrinker. Indeed,

$$S = 0$$

is exactly the self-shrinker equation.

Moreover, along rescaled mean curvature flow,

$$(\partial_s x)^\perp = S,$$

while the first variation of Gaussian area is

$$DF_\Sigma(V) = -\int_\Sigma \langle S, V\rangle \, \rho \, d\mu.$$

Thus

$$\nabla_{L^2_\rho} F = -S.$$

In this sense, S is simultaneously the shrinker defect, the normal velocity of rescaled mean curvature flow, and minus the gradient of the Gaussian area.

6.7 Łojasiewicz inequalities for cylindrical singularities

The round self-shrinking cylinders in $\mathbb{R}^{n+1}$ are

$$\mathcal{C}_k = S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}, \qquad 1 \le k \le n.$$

Let $\mathfrak{C}_k$ denote the family obtained from $\mathcal{C}_k$ by rotating its axis.

For closed embedded surfaces in $\mathbb{R}^3$, Huisken's genericity conjecture has been resolved: after an arbitrarily small generic perturbation, the flow encounters only spherical and cylindrical singularities. In arbitrary dimensions, round spheres, cylinders, and planes are the only entropy-stable smooth self-shrinkers, although a completely unrestricted global perturbation theorem in every dimension requires additional qualifications.

The standard Łojasiewicz–Simon theorem for compact critical points cannot be applied directly to a cylinder because $\mathcal{C}_k$ is noncompact. Even if a rescaled flow converges locally to a cylinder, the evolving hypersurface need not be a global normal graph over the entire cylinder.

Colding and Minicozzi therefore prove localized inequalities on large balls, with exponentially small errors arising from cutoff terms.

Theorem 6.5 (Colding–Minicozzi, schematic form)**.** *Suppose that a hypersurface Σ has controlled entropy and geometry and is sufficiently close, on a large ball, to a multiplicity-one round cylinder in $\mathfrak{C}_k$. Then there are localized distance and gradient inequalities.*

The precise distance inequality has the form

$$d^2_{\mathfrak{C}_k}(R) \le CR^\rho \left(||\phi||^{b_{\ell,n}}_{L^1(B_R)} + e^{-b_{\ell,n} R^2/4} \right),$$

where

$$\phi = \frac{\langle x, \nu\rangle}{2} - H$$

is, up to sign, the scalar shrinker defect, and

$$0 < b_{\ell,n} < 1, \qquad b_{\ell,n} \to 1 \quad \textit{as } \ell \to \infty.$$

The precise gradient inequality similarly bounds

$$|F(\Sigma) - F(\mathcal{C}_k)|$$

by a power of a localized norm of ϕ, together with exponentially decaying cutoff errors.

When these estimates are applied along a rescaled mean curvature flow, the cutoff errors and localized terms can be controlled. Schematically, one obtains

$$d_{\mathfrak{C}_k}(\Sigma_s)^2 \lesssim ||S||_{L^2_\rho(\Sigma_s)}$$

and

$$|F(\Sigma_s) - F(\mathcal{C}_k)|^{2/3} \lesssim ||S||_{L^2_\rho(\Sigma_s)}.$$

The second estimate is a gradient Łojasiewicz inequality with

$$1 - \theta = \frac{2}{3}, \qquad \theta = \frac{1}{3}.$$

These inequalities imply that the rescaled flow has finite total motion relative to the family of cylinders. In particular, the axis of the best-fitting cylinder cannot continue to rotate indefinitely.

Theorem 6.6 (Colding–Minicozzi)**.** *If one tangent flow at a singular point is a multiplicity-one round cylinder, then the tangent flow is unique. In particular, every subsequential tangent flow has the same cylindrical axis.*

6.8 The arrival-time equation

Suppose (M_t) is a mean-convex flow sweeping through a region Ω. Define the **arrival-time function**

$$u : \Omega \to \mathbb{R}$$

by declaring that

$$M_t = \{x \in \Omega : u(x) = t\}.$$

At points where $\nabla u \neq 0$, the unit normal to the level set is, up to orientation,

$$\nu = \frac{\nabla u}{|\nabla u|}.$$

The arrival-time function satisfies the degenerate elliptic equation

$$|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = -1.$$

The equation degenerates at the critical points of u, which are closely related to the singular set of the mean curvature flow.

7 Applications to Ricci-flat manifolds: uniqueness of tangent cones

Let (M^n, g, p) be a complete noncompact Ricci-flat manifold with $n \geq 3$. By Bishop–Gromov comparison, the asymptotic volume ratio

$$\mathrm{AVR}(M) := \lim_{r\to\infty} \frac{\mathrm{Vol}(B_r(p))}{\omega_n r^n}$$

exists and is nonnegative.

We assume that M has **Euclidean volume growth**, meaning that

$$\mathrm{AVR}(M) > 0.$$

7.1 The Green function and the conical defect

Let $G = G(p, \cdot)$ be the normalized minimal positive Green function with pole at p, chosen so that

$$\Delta G = -n(n-2)\omega_n \delta_p.$$

With this normalization, on Euclidean space

$$G(p, x) = |x - p|^{2-n}.$$

Define

$$b = G^{1/(2-n)}.$$

On Euclidean space,

$$b(x) = |x - p|.$$

Away from the pole, $G = b^{2-n}$ is harmonic, and a calculation gives

$$\Delta b^2 = 2n\,|\nabla b|^2.$$

On Euclidean space,

$$\operatorname{Hess} b^2 = 2g.$$

More generally, define the trace-free Hessian tensor

$$B := \operatorname{Hess} b^2 - \frac{\Delta b^2}{n} g = \operatorname{Hess} b^2 - 2\,|\nabla b|^2\, g.$$

The tensor B is a **conical defect**: it vanishes on an exact metric cone, where b is proportional to the radial coordinate.

Define

$$A(r) = r^{1-n} \int_{\{b=r\}} |\nabla b|^3 \, d\mu.$$

For a Ricci-flat manifold, the Green-function monotonicity formula gives

$$A'(r) = -\frac{1}{2} r^{n-3} \int_{\{b\geq r\}} b^{2-2n}\, |B|^2 \, d\mu \leq 0.$$

Thus $A(r)$ is monotone nonincreasing and has a limit

$$A_\infty = \lim_{r\to\infty} A(r).$$

On a metric cone, $B = 0$ and A is constant.

This is an elliptic analogue of Huisken's monotonicity formula:

$$\text{drop of } A = \text{integrated square of the conical defect.}$$

Define the scale-invariant tail defect

$$Q(r) = \int_{\{b\geq r\}} b^{-n} |B|^2 \, d\mu.$$

The function Q is monotone nonincreasing and roughly measures $-rA'(r)$. Moreover,

$$Q(R/2) - Q(8R) = \int_{\{R/2\leq b\leq 8R\}} b^{-n} |B|^2 \, d\mu.$$

7.2 Distance to metric cones

Let $C(Z)$ denote the metric cone over a compact metric space Z. Define the scale-invariant distance to the nearest cone by

$$\Theta_R \coloneqq \inf_{C(Z)} R^{-1} d_{\mathrm{GH}} \left(B_p(4R) \setminus B_p(R), B_0^{C(Z)}(4R) \setminus B_0^{C(Z)}(R) \right).$$

More precise definitions include a harmless normalization relating the Green-function radius b to the distance from p.

For every $\mu > 0$, one has an estimate of the form

$$\Theta_R^{2+\mu} \leq C_\mu \int_{\{R/2\leq b\leq 8R\}} b^{-n} |B|^2 \, d\mu.$$

Equivalently,

$$\Theta_R^{2+\mu} \leq C_\mu \big(Q(R/2) - Q(8R) \big).$$

Thus a small drop of Q implies that the annulus at scale R is close to a metric cone.

7.3 Tangent cones at infinity

Let $r_j \to \infty$. By pointed Gromov compactness, after passing to a subsequence,

$$(M, r_j^{-2} g, p) \longrightarrow (X, d_X, x_\infty)$$

in the pointed Gromov–Hausdorff topology.

Such a subsequential limit is called a **tangent cone at infinity**. Under nonnegative Ricci curvature and Euclidean volume growth, every tangent cone at infinity is a metric cone

$$C(Y).$$

A priori, the cone may depend on the sequence r_j.

Theorem 7.1 (Colding–Minicozzi)**.** *Let (M^n, g) be a complete Ricci-flat manifold with Euclidean volume growth. If one tangent cone at infinity has a smooth cross-section, then the tangent cone at infinity is unique.*

The uniqueness argument is based on proving

$$\sum_{j=1}^{\infty} \Theta_{2^j} < \infty.$$

Indeed, Θ_{2^j} controls the change between the best approximating cones at consecutive dyadic scales. Summability therefore implies that these approximating cones form a Cauchy sequence.

This sum is analogous to the length of a gradient-flow trajectory:

$$\sum_j \Theta_{2^j} \quad \text{plays the role of} \quad \int_0^\infty |x'(t)|\, dt.$$

7.4 The auxiliary analytic functional

There is no literal gradient flow in this problem. Instead, the scale R plays the role of time, and the level-set data

$$\Big(R^{-2} g_{\{b=R\}}, |\nabla b|_{\{b=R\}}\Big)$$

trace a curve in a space of metrics and positive weights on the cross-section.

The monotone quantity $A(R)$ is geometrically natural, but it is not itself an analytic functional on a fixed Banach space. Colding and Minicozzi therefore introduce an analytic weighted Einstein–Hilbert-type functional

$$\mathcal{R}(g, w)$$

on pairs consisting of a metric g and a positive weight w, subject to a fixed weighted-volume constraint.

At the limiting cone data (g_∞, w_∞),

$$\nabla_1 \mathcal{R}(g_\infty, w_\infty) = 0, \qquad \mathcal{R}(g_\infty, w_\infty) = A_\infty.$$

Here ∇_1 denotes the gradient tangent to the weighted-volume constraint.

After restricting to a slice for the diffeomorphism action, the functional satisfies a Łojasiewicz–Simon inequality: for some $\alpha < 1$,

$$|\mathcal{R}(g, w) - \mathcal{R}(g_\infty, w_\infty)|^{2-\alpha} \le C \, ||\nabla_1 \mathcal{R}(g, w)||^2 .$$

The level-set data satisfy the comparison estimates

$$\big|\big|\nabla_1 \mathcal{R}\left(R^{-2} g_{\{b=R\}}, |\nabla b|\right)\big|\big|^2 \le C\big(Q(R/2) - Q(2R)\big),$$

and, schematically,

$$A(R) - A_\infty \lesssim \left|\mathcal{R}\left(R^{-2} g_{\{b=R\}}, |\nabla b|\right) - \mathcal{R}(g_\infty, w_\infty)\right| + Q(R/2) - Q(2R).$$

Combining these estimates with the Łojasiewicz–Simon inequality gives a discrete gradient inequality

$$Q(2R)^{2-\alpha} \le C\big(Q(R/2) - Q(2R)\big).$$

Iteration yields, for some $\epsilon > 0$,

$$Q(R) \leq \frac{C}{(\log R)^{1+\epsilon}}.$$

Together with the distance estimate for Θ_R, this implies

$$\sum_j \Theta_{2^j} < \infty$$

and hence uniqueness of the tangent cone.

Remark 7.2. The diffeomorphism invariance of $\mathcal{R}$ initially gives an infinite-dimensional kernel in its Hessian. An Ebin–Palais slice is used to remove these gauge directions. On the slice, the remaining kernel is finite-dimensional, so a Lyapunov–Schmidt reduction can be used to prove the Łojasiewicz–Simon inequality.

This is the same symmetry-removal mechanism encountered earlier for geometric variational problems.

Remark 7.3. Earlier work of Cheeger–Tian proved uniqueness under integrability and additional regularity assumptions on the tangent cones. The Colding–Minicozzi theorem removes the integrability assumption: it is enough that one tangent cone has a smooth cross-section.

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